Rindler spacetime, introduction to the Unruh-Fulling (UF) effect

Ashmita Das*
Department of Physics, School of Engineering and Sciences, SRM University AP

Contents

I. Rindler wedges: brief discussion

Motivation: we try to understand how the result of fundamental Physics depends on the perspective of the observer. In GR the observer dependence plays key role. Moreover when we include quantum field theory within curved spacetime, observer dependence comes into the picture to decide the vacuum state. This is the building block to understand the particle production in curved spacetime.

At first we will try to understand how observer dependence affects the physical outcome in flat spacetime background. We start with the the covariant formulation where the timelike worldline of a massive particle is parametrised by proper time, $x^a = x^a(\tau)$. The velocity (tangent) vector can be written as:

$$u^a = \frac{dx^a}{d\tau} \tag{1}$$

and it is a Lorentz vector, normalised as [note metric signature as (+,-,-,-)],

$$u^a u_a = \eta_{ab}\, u^a\, u^b = c^2 = 1 \quad \text{(in natural unit we take c = 1)} \tag{2}$$

The Lorentz covariant acceleration is the 4-vector,

$$a^c = \frac{d}{d\tau} u^c = \frac{d^2}{d\tau^2} x^c \tag{3}$$

and the equation of motion of a massive free particle becomes,

$$a^c = \frac{d^2}{d\tau^2} x^c(\tau) = 0 \tag{4}$$

Differentiating Eq. (2) for non zero acceleration, we obtain,

$$a^c u_c = \eta_{cb}\, a^c\, u^b = 0 \tag{5}$$

and therefore spacelike, $\eta_{cb}\, a^c\, a^b = -a^2 < 0$.
Using Eq. (2), let us take that an observer is accelerating in the direction x^1 direction (so that in the momentary rest frame of this observer one has $u^a = (1, 0, 0, 0)$, $a^a = (0, -a, 0, 0)$), we will say that the observer undergoes constant acceleration if a is time independent. To determine the worldline of such an observer, we note that the general solution to Eq. (2) with $u^2 = u^3 = 0$,

$$\eta_{ab} u^a u^b = (u^0)^2 - (u^1)^2 = 1, \quad \text{is,} \qquad u^0 = \cosh F(\tau), \quad u^1 = \sinh F(\tau) \tag{6}$$

for some function $F(\tau)$.

* ashmita.d@srmap.edu.in

- Find the acceleration from u^0 and u^1.
- Also find the norm of the acceleration vector $a^2 = \dot{F}^2$. What will be $F(\tau)$ for an observer with constant acceleration?
- After obtaining $F(\tau)$ show that: $u^a(\tau) = (\cosh a\tau,\, \sinh a\tau,\, 0,\, 0)$ and $x^a(\tau) = \left(a^{-1}\sinh a\tau,\, a^{-1}\cosh a\tau,\, 0,\, 0\right)$. Note $x^a(\tau)$ is the worldline of an observer with constant acceleration a.
- Show that the worldline of this observer is the hyperbola $\eta_{ab}\, x^a\, x^b = -(x^0)^2 + (x^1)^2 = a^{-2}$ in the quadrant $x^1 > |x^0|$ of Minkowski spacetime. Note: these trajectories for different values of a is confined to the region $x^1 > |x^0|$ and it is known as Right Rindler Wedge (RRW).

Thus we get the coordinate transformation for the uniformly accelerated observer as:

$$t = \frac{1}{a}\sinh(a\tau) \qquad x = t = \frac{1}{a}\cosh(a\tau) \quad y = z = 0 \tag{7}$$

and we also suppress the transverse dimensions $(y,\, z)$ of the spectator.

- Considering the transformation:

$$t = a^{-1}\, e^{a\xi}\sinh(a\tau), \qquad\qquad x = a^{-1}\, e^{a\xi}\cosh(a\tau)\ , \tag{8}$$

  find how the Minkowski line element ds^2 changes? you may take $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ You can also realise that for the proper frame condition in the above transformation i.e $\xi = 0$, you will achieve the transformation for uniformly accelerated trajectory as in Eq. (7).

As similar to the above discussion, the coordinates $\left(\bar{\tau},\, \bar{\xi}\right)$ represent the Left Rindler Wedge (LRW) are given by,

$$t = a^{-1}\, e^{a\bar{\xi}}\sinh(a\bar{\tau}), \qquad\qquad z = \,-\,a^{-1}\, e^{a\bar{\xi}}\cosh(a\bar{\tau})\ . \tag{9}$$

- Find the hyperbola trajectory for LRW.

- There is no common solution of the two equations $t = \pm x$ and $x^2 - t^2 = a^{-2}$, thus $t = \pm x$ represents a horizon for the trajectory $x^2 - t^2 = a^{-2}$ [see the diagramme].
- Also you can realise that due to the presence of the horizon, two wedges become causally disconnected.
- If we use the transformation $t = T\cosh(a\zeta),\ x = T\sinh(a\zeta)$ covers the region $t > |x|$. This mimics the expanding kasner Universe and one can also write what would be the transformation for the contracting Kasner Universe. [see fig. (1)].

II. A brief introduction to the use of quantum field theory in understanding UF effect

For simplicity we consider $(D+1)$ dimensional spacetime whose metric takes the form:

$$ds^2 = [N(x)]^2\, dt^2 - G_{ab}(x)\, dx^a\, dx^b \tag{10}$$

The coefficient $N(x)$ is called the lapse function and G_{ab} is the metric on the spacelike hypersurface of constant t. In this spacetime the minimally coupled massive Klein-Gordon (KG) equation $\left(\nabla_\mu \nabla^\mu + m^2\right)\phi = 0$,which arises as the Euler-Lagrange (EL) equation from the Lagrangian density,

$$\mathcal{L} = \frac{\sqrt{-g}}{2}\left(\nabla_\mu\phi\, \nabla^\mu\phi - m^2\phi^2\right) \tag{11}$$

- Starting from either variational principle or using EL equation find the equation of motion for the scalar field? Hint: you may take $\left[\nabla_\mu\nabla^\mu\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right)\right]$. Remember, the space indices a and b run from 1 to D.

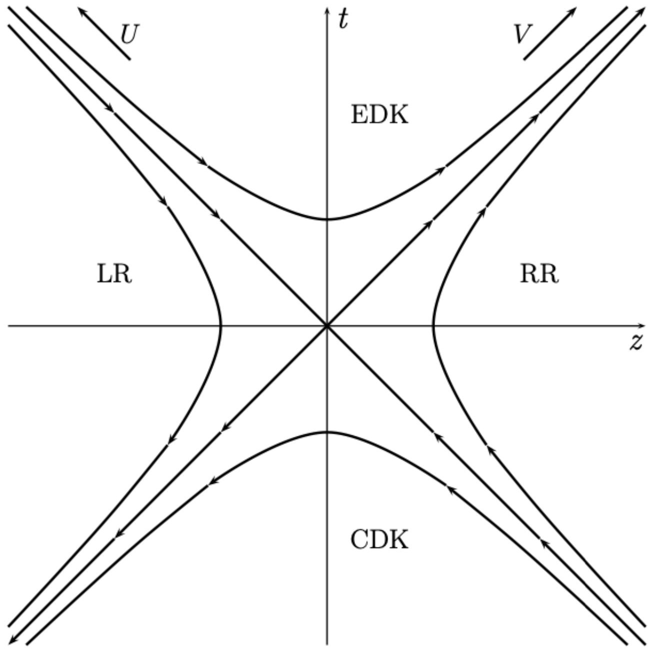

FIG. 1: The regions with $|t| < z$, $|t| < -z$, $t > |z|$ and $t < -|z|$, denoted RR, LR, EDK and CDK, respectively, are the left Rindler wedge, right Rindler wedge, expanding degenerate Kasner universe and contracting degenerate Kasner universe, respectively.

Given two complex solutions $f_A(x)$ and $f_B(x)$ to the KG equation, we define the KG current such as,

$$J^{\mu}_{(f_A, f_B)}(x) \equiv f_A^*(x)\nabla^{\mu} f_B(x) - f_B(x)\nabla^{\mu} f_A^*(x) \tag{12}$$

Recall: the scalar product in QM can be written as $(\psi, \phi) = \psi^*\phi$ while $(\phi, \psi) = \phi^*\psi$. Moreover, $(\psi, \phi) = \int \psi^*(x)\phi(x)\, dx$.

- Show that $\nabla_{\mu} J^{\mu}_{(f_A, f_B)}(x) = 0$.

Recall from our knowledge of quantum mechanics (QM): Probability density: $\rho(\mathbf{x}, t) = \psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)$ and $\mathbf{J}(\mathbf{x}, t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi\right)$ In QM we get,

$$\frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \nabla.\mathbf{J} = 0 \tag{13}$$

- Show the continuity condition from the time dependent Schrödinger equation.

A. Understanding inner products in QFT

In QFT we define the KG inner product as follows:

$$(f_A,\, f_B)_{\mathrm{KG}} \equiv i \int d^D x \sqrt{G}\, n_{\mu}\, J^{\mu}_{(f_A, f_B)} \tag{14}$$

is independent of time t and n_{μ} is the future directed unit vector normal to the hypersurface Σ of constant t. for this metric the above equation becomes,

$$(f_A,\, f_B)_{\mathrm{KG}} = i \int d^D x \sqrt{G}\, N^{-1} \left(f_A^* \partial_t f_B - f_B \partial_t f_A^*\right) \tag{15}$$

Please verify the above equation. In doing this remember the surface Σ_t is define by the $t=$ constant, thus the normal to this surface can be written as $n_\mu = A\partial_\mu t = A(1,0,0,0)$ Now requiring $n_\mu n_\mu = 1$, one obtains, $A = N$ for this metric.
The conjugate momentum density $\pi(x)$ is defined as $\pi \equiv \partial\mathcal{L}/\partial\dot{\phi}$, where $\dot{\phi} \equiv \partial_t\phi$. For the metric (2.1) one finds

$$\pi(x) = N^{-1}\sqrt{G}\,\dot{\phi}(x). \quad \text{(verify)} \tag{16}$$

Note that, if we let $p_A(x)$ and $p_B(x)$ be the conjugate momentum density for the solutions $f_A(x)$ and $f_B(x)$, respectively, then the Klein-Gordon inner product can be expressed as

$$(f_A, f_B)_{\rm KG} = i\int d^D\mathbf{x}\,[f_A^*(x)p_B(x) - p_A^*(x)f_B(x)]\,. \tag{17}$$

B. Quantisation cont..

The quantization of the field ϕ proceeds as follows. We denote the field operators corresponding to ϕ and π by $\hat{\phi}$ and $\hat{\pi}$, respectively. One imposes the following equal time canonical commutation relations: [please recall your canonical commutation relation in QM which has been promoted to now field]

$$\left[\hat{\phi}(t,\mathbf{x}), \hat{\phi}(t,\mathbf{x}')\right] = [\hat{\pi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{x}')] = 0,$$

$$\left[\hat{\phi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{x}')\right] = i\delta^D(\mathbf{x},\mathbf{x}'),$$

where the delta function $\delta^D(\mathbf{x},\mathbf{x}')$ is defined by

$$\int d^D\mathbf{x}\, f(\mathbf{x})\delta^D(\mathbf{x},\mathbf{x}') = f(\mathbf{x}').$$

For arbitrary complex valued solutions $f_A(x)$ and $f_B(x)$ to the Klein-Gordon equation (with a suitable integrability conditions) one obtains,

$$\left[(f_A, \hat{\phi})_{\rm KG},\, (\hat{\phi}, f_B)_{\rm KG}\right] = (f_A, f_B)_{\rm KG} \tag{18}$$

Now, assume that there is a complete set of solutions, $\{f_i, f_i^*\}$, to the Klein-Gordon equation

$$\partial_t\left(N^{-1}\sqrt{G}\,\partial_t\phi\right) - \partial_a\left(N\sqrt{G}\,G^{ab}\partial_b\phi\right) + N\sqrt{G}\,m^2\phi = 0 \tag{19}$$

satisfying

$$(f_i, f_j)_{\rm KG} = -(f_i^*, f_j^*)_{\rm KG} = \delta_{ij}, \tag{20}$$

$$(f_i^*, f_j)_{\rm KG} = (f_i, f_j^*)_{\rm KG} = 0. \tag{21}$$

We assume here that the indices labeling the solutions are discrete for simplicity of the discussion but its extension to the cases with continuous labels is straightforward. In Minkowski spacetime one chooses the positive-frequency modes as f_i's and, consequently, the negative-frequency modes as f_i^*'s.
Expanding the quantum field $\hat{\phi}(x)$ as

$$\hat{\phi}(x) = \sum_i\left[\hat{a}_i f_i(x) + \hat{a}_i^\dagger f_i^*(x)\right], \tag{22}$$

one finds

$$\hat{a}_i = (f_i, \hat{\phi})_{\rm KG}, \quad \hat{a}_i^\dagger = (\hat{\phi}, f_i)_{\rm KG}. \tag{23}$$

- Try to show, by using Eqs. (18), (20), and (21), that

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \quad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}. \tag{24}$$

- Important note: The operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ are called the *annihilation and creation operators*, respectively. The vacuum state $|0\rangle$ is defined by requiring $\hat{a}_i|0\rangle = 0$. The Fock space of states is obtained by applying the creation operators $\hat{a}_i^\dagger$ on the vacuum state $|0\rangle$. We call the operator $\hat{N}_i = \hat{a}_i^\dagger \hat{a}_i$ (with no summation on the right-hand side) the *number operator* in the mode 'i'. However, note that, since it is not always easy to construct a (theoretical) detector model which is excited when the eigenvalue of $\hat{N}_i$ changes from 1 to 0, say, the operator $\hat{N}_i$ does not necessarily lead to a useful particle concept.

- Since the coefficient operators $\hat{a}_i$ of the functions f_i annihilate the vacuum state $|0\rangle$, the choice of the functions f_i satisfying Eqs. (20) and (21) determines the vacuum state. For this reason we call the functions f_i the *positive-frequency modes* and their complex conjugates f_i^* the *negative-frequency modes* in analogy with the case in Minkowski spacetime. Thus, the choice of the positive-frequency modes determines the vacuum state. In a general curved spacetime there is no privileged choice of the positive-frequency modes, and consequently, there is no privileged vacuum state unlike in Minkowski spacetime, as we mentioned before.

C. Brief idea of the Bogoliubov Coefficients

Now, suppose that two complete sets of positive frequency modes $\{f_i^{(1)}\}$ and $\{f_I^{(2)}\}$ satisfy the Klein-Gordon inner-product relations (20) and (21), where the lower-case letters i, j are replaced by the upper-case equivalents I, J for $f_I^{(2)}$. Since both sets are complete, the modes $f_I^{(2)}$ can be expressed as linear combinations of $f_i^{(1)}$ and $f_i^{(1)*}$, and vice versa. Thus,

$$f_I^{(2)} = \sum_i \left[\alpha_{Ii} f_i^{(1)} + \beta_{Ii} f_i^{(1)*}\right], \tag{25}$$

$$f_I^{(2)*} = \sum_i \left[\alpha_{Ii}^* f_i^{(1)*} + \beta_{Ii}^* f_i^{(1)}\right]. \tag{26}$$

By noting that [Can be easily verified by using the definition of inner product. Please check]

$$\alpha_{Ii} = \left(f_i^{(1)}, f_I^{(2)}\right)_{\text{KG}} = \left(f_I^{(2)}, f_i^{(1)}\right)_{\text{KG}}^*, \tag{27}$$

$$\beta_{Ii} = -\left(f_i^{(1)*}, f_I^{(2)}\right)_{\text{KG}} = \left(f_I^{(2)*}, f_i^{(1)}\right)_{\text{KG}}, \tag{28}$$

one can express $f_i^{(1)}$ as a linear combination of $f_I^{(2)}$ and $f_I^{(2)*}$ as

$$f_i^{(1)} = \sum_I \left[\alpha_{Ii}^* f_I^{(2)} - \beta_{Ii} f_I^{(2)*}\right], \tag{29}$$

$$f_i^{(1)*} = \sum_I \left[\alpha_{Ii} f_I^{(2)*} - \beta_{Ii}^* f_I^{(2)}\right]. \tag{30}$$

The scalar field $\hat{\phi}(x)$ can be expanded using either of the two sets $\{f_i^{(1)}\}$ and $\{f_I^{(2)}\}$:

$$\begin{aligned}\hat{\phi}(x) &= \sum_i \left[\hat{a}_i^{(1)} f_i^{(1)} + \hat{a}_i^{(1)\dagger} f_i^{(1)*}\right] \\ &= \sum_I \left[\hat{a}_I^{(2)} f_I^{(2)} + \hat{a}_I^{(2)\dagger} f_I^{(2)*}\right].\end{aligned} \tag{31}$$

Using the expansion given by Eqs. (25) and (26), and comparing the coefficients of $f_i^{(1)}$ and $f_i^{(1)*}$, we find

$$\hat{a}_i^{(1)} = \sum_I \left(\alpha_{Ii} \hat{a}_I^{(2)} + \beta_{Ii}^* \hat{a}_I^{(2)\dagger}\right). \tag{32}$$

and similarly by using Eqs. (29) and (30) we have

$$\hat{a}_I^{(2)} = \sum_i \left(\alpha_{Ii}^* \hat{a}_i^{(1)} - \beta_{Ii}^* \hat{a}_i^{(1)\dagger} \right). \tag{33}$$

This transformation, which mixes annihilation and creation operators, is called a *Bogolubov transformation*, and the coefficients α_{Ii} and β_{Ii} are called the *Bogolubov coefficients*. The Bogolubov transformation found its first major application to QFT in curved spacetime in the derivation of particle creation in expanding universes.
The vacuum states $|0_{(1)}\rangle$ and $|0_{(2)}\rangle$ corresponding to the two sets of positive-frequency modes $\{f_i^{(1)}\}$ and $\{f_I^{(2)}\}$, respectively, are distinct if β_{Ii} do not vanish for all I and i. For example, the expectation value of the number operator

$$\hat{N}_i^{(1)} = \hat{a}_i^{(1)\dagger} \hat{a}_i^{(1)}$$

for the state $|0_{(1)}\rangle$ is zero by definition but for the state $|0_{(2)}\rangle$ it can be calculated by using Eq. (32) as,

$$\langle 0_{(2)} | \hat{N}_i^{(1)} | 0_{(2)} \rangle = \sum_I |\beta_{Ii}|^2. \tag{34}$$

We similarly find for the number operator $\hat{N}_I^{(2)} = \hat{a}_I^{(2)\dagger} \hat{a}_I^{(2)}$,

$$\langle 0_{(1)} | \hat{N}_I^{(2)} | 0_{(1)} \rangle = \sum_i |\beta_{Ii}|^2. \tag{35}$$